The Stability of Linear Feedback Systems

The issue of ensuring the stability of a closed-loop feedback system is central to control system design. Knowing that an unstable closed-loop system is generally of no practical value, we seek methods to help us analyze and design stable systems. A stable system should exhibit a bounded output if the corresponding input is bounded. This is known as bounded-input, bounded-output stability and is one of the main topics of this chapter.

The stability of a feedback system is directly related to the location of the roots of the characteristic equation of the system transfer function. The Routh–Hurwitz method is introduced as a useful tool for assessing system stability. The technique allows us to compute the number of roots of the characteristic equation in the right half-plane without actually computing the values of the roots. Thus we can determine stability without the added computational burden of determining characteristic root locations. This gives us a design method for determining values of certain system parameters that will lead to closed-loop stability. For stable systems we will introduce the notion of relative stability, which allows us to characterize the degree of stability.

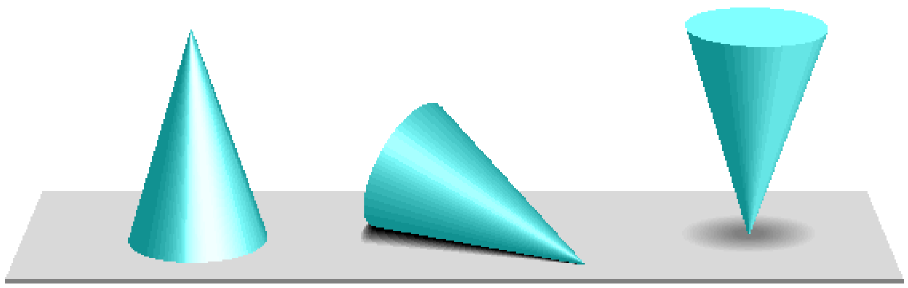
The Concept of Stability

A stable system is a dynamic system with a bounded response to a bounded input.

Absolute stability is a stable/not stable characterization for a closed-loop feedback system. Given that a system is stable we can further characterize the degree of stability, or the relative stability.

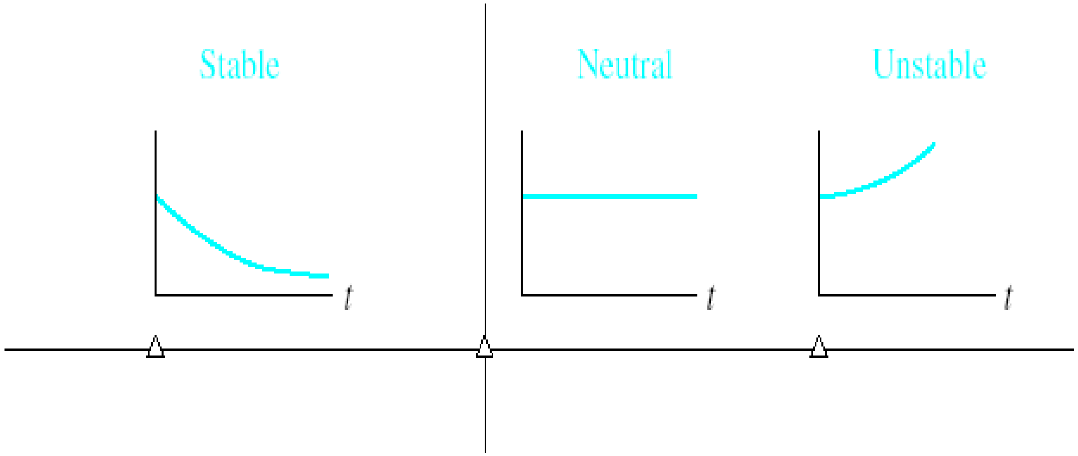
The Concept of Stability

The concept of stability can be illustrated by a cone placed on a plane horizontal surface.



(a) Stable (b) Neutral (c) Unstable

A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.



A system is considered marginally stable if only certain bounded inputs will result in a bounded output.

The Routh-Hurwitz Stability Criterion

It was discovered that all coefficients of the characteristic polynomial must have the same sign and non-zero if all the roots are in the left-hand plane.

These requirements are necessary but not sufficient. If the above requirements are not met, it is known that the system is unstable. But, if the requirements are met, we still must investigate the system further to determine the stability of the system.

The Routh-Hurwitz criterion is a necessary and sufficient criterion for the stability of linear systems.

The Routh-Hurwitz Stability Criterion

Characteristic equation, q(s) → $a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \Lambda + a_1 s + a_0 = 0$

Routh array →

$$
\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
s^0 & h_{n-1} & &
\end{array}
$$

The Routh-Hurwitz criterion states that the number of roots of q(s) with positive real parts is equal to the number of changes in sign of the first column of the Routh array.

$$b_{n-1} = \frac{(a_{n-1})(a_{n-2}) - a_n(a_{n-3})}{a_{n-1}} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_{n-3} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_{n-2} & a_{n-4} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}$$

The Routh-Hurwitz Stability Criterion

Case One: No element in the first column is zero.

Example 6.1   Second-order system

The Characteristic polynomial of a second-order system is:

$$q(s) = a_2 \cdot s^2 + a_1 \cdot s + a_0$$

The Routh array is written as:

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & b_1 & 0 \end{array}$$

where:

$$b_1 = \frac{a_1 \cdot a_0 - (0) \cdot a_2}{a_1} = a_0$$

Therefore the requirement for a stable second-order system is simply that all coefficients be positive or all the coefficients be negative.

The Routh-Hurwitz Stability Criterion

Case Two: Zeros in the first column while some elements of the row containing a zero in the first column are nonzero.

If only one element in the array is zero, it may be replaced with a small positive number ε that is allowed to approach zero after completing the array.

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

The Routh array is then:

s^5	1	2	11
s^4	2	4	10
s^3	b_1	6	0
s^2	c_1	10	0
s^1	d_1	0	0
s^0	10	0	0

where:

$$b_1 = \frac{2\cdot 2 - 1\cdot 4}{2} = 0 = \varepsilon \qquad c_1 = \frac{4\varepsilon - 2\cdot 6}{\varepsilon} = \frac{-12}{\varepsilon} \qquad d_1 = \frac{6\cdot c_1 - 10\varepsilon}{c_1} = 6$$

There are two sign changes in the first column due to the large negative number calculated for c1. Thus, the system is unstable because two roots lie in the right half of the plane.

The Routh-Hurwitz Stability Criterion

Case Three: Zeros in the first column, and the other elements of the row containing the zero are also zero.

This case occurs when the polynomial q(s) has zeros located symetrically about the origin of the s-plane, such as $(s+\sigma)(s-\sigma)$ or $(s+j\omega)(s-j\omega)$. This case is solved using the auxiliary polynomial, U(s), which is located in the row above the row containing the zero entry in the Routh array.

$q(s) = s^3 + 2\cdot s^2 + 4s + K$

Routh array:

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & K \\ s^1 & \frac{8-K}{2} & 0 \\ s^0 & K & 0 \end{array}$$

For a stable system we require that $0 < s < 8$

For the marginally stable case, K=8, the s^1 row of the Routh array contains all zeros. The auxiliary plynomial comes from the s^2 row.

$U(s) = 2s^2 + Ks^0 = 2\cdot s^2 + 8 = 2\left(s^2 + 4\right) = 2(s + j\cdot 2)(s - j\cdot 2)$

It can be proven that U(s) is a factor of the characteristic polynomial:

$\frac{q(s)}{U(s)} = \frac{s + 2}{2}$

Thus, when K=8, the factors of the characteristic polynomial are:

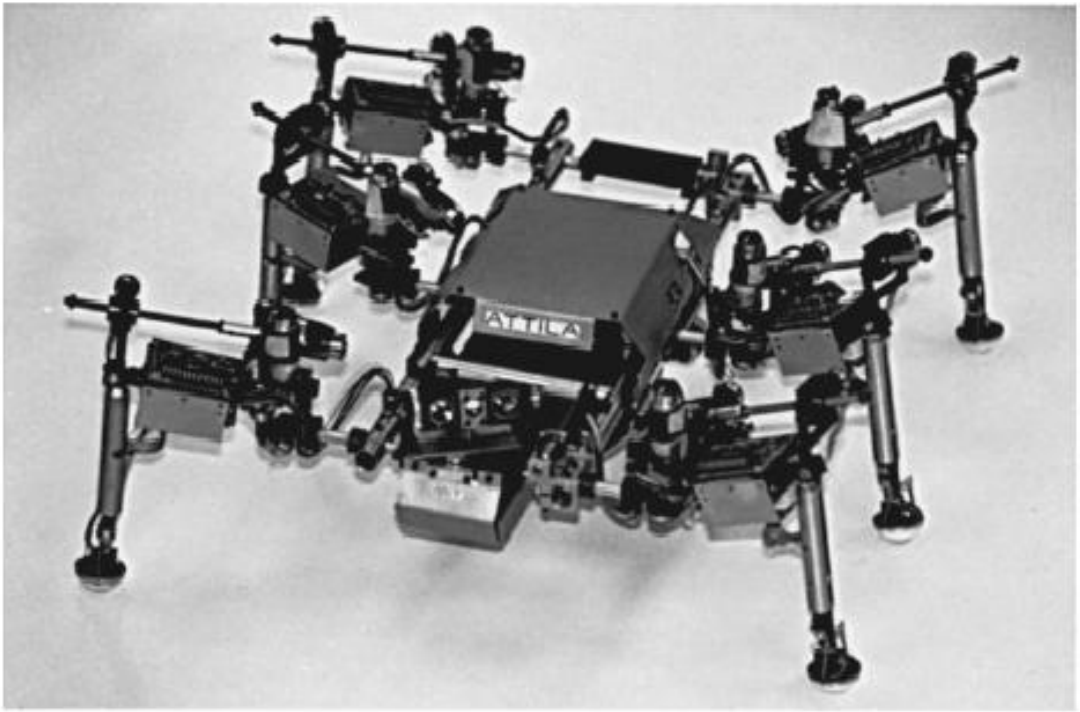
$q(s) = (s + 2)(s + j\cdot 2)(s - j\cdot 2)$

The Routh-Hurwitz Stability Criterion

Case Four: Repeated roots of the characteristic equation on the jw-axis.

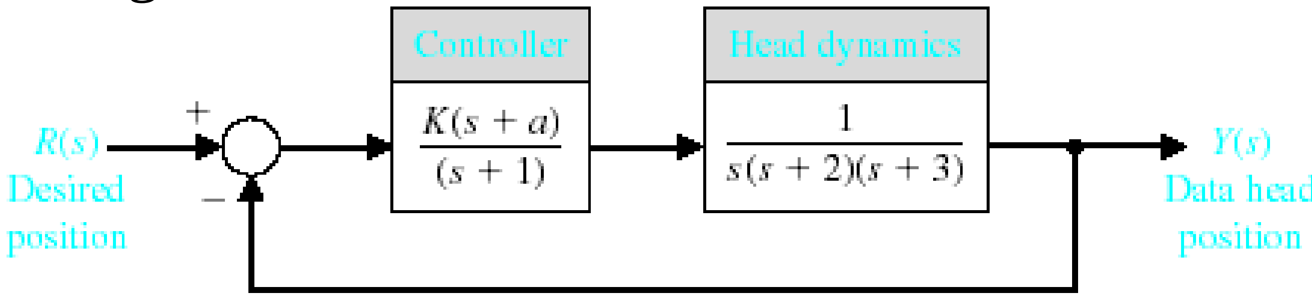
With simple roots on the jw-axis, the system will have a marginally stable behavior. This is not the case if the roots are repeated. Repeated roots on the jw-axis will cause the system to be unstable. Unfortunately, the routh-array will fail to reveal this instability.

Example



A completely integrated, six-legged, micro robot system. The six-legged design provides maximum dexterity. Legs also provide a unique sensory system for environmental interaction. It is equipped with a sensor network that includes 150 sensors of 12 different types. The legs are instrumented so that the robot can determine the lay of the terrain, the surface texture, hardness, and even color. The gyro-stabilized camera and range finder can be used for gathering data beyond the robot's immediate reach. This high-performance system is able to walk quickly, climb over obstacles, and perform dynamic motions. (Courtesy of IS Robotics Corporation.)

Example : Welding control



Welding head position control.

Using block diagram reduction we find that: $q(s) = s^4 + 6s^3 + 11s^2 + (K+6)s + Ka$

The Routh array is then:

$$\begin{array}{c|ccc} s^4 & 1 & 11 & Ka \\ s^3 & 6 & (K+6) & \\ s^2 & b_3 & Ka & \\ s^1 & c_3 & & \\ s^0 & Ka & & \end{array}$$

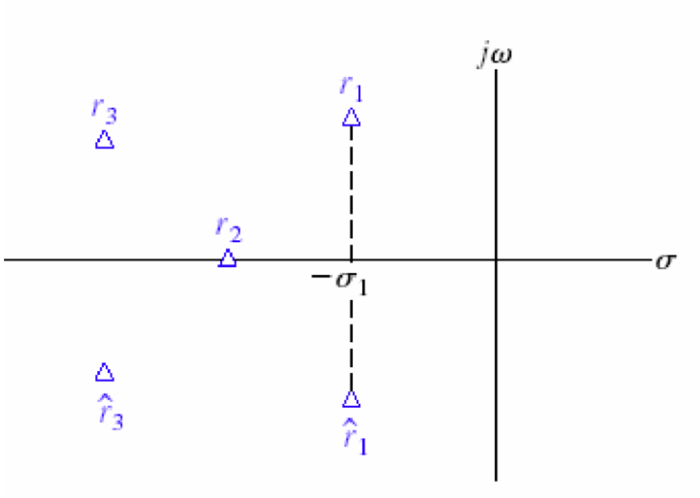
where: $b_3 = \frac{60-K}{6}$ and $c_3 = \frac{b_3(K+6) - 6 \cdot Ka}{b_3}$

For the system to be stable both b_3 and c_3 must be positive.

Using these equations a relationship can be determined for K ar

The Relative Stability of Feedback Control Systems

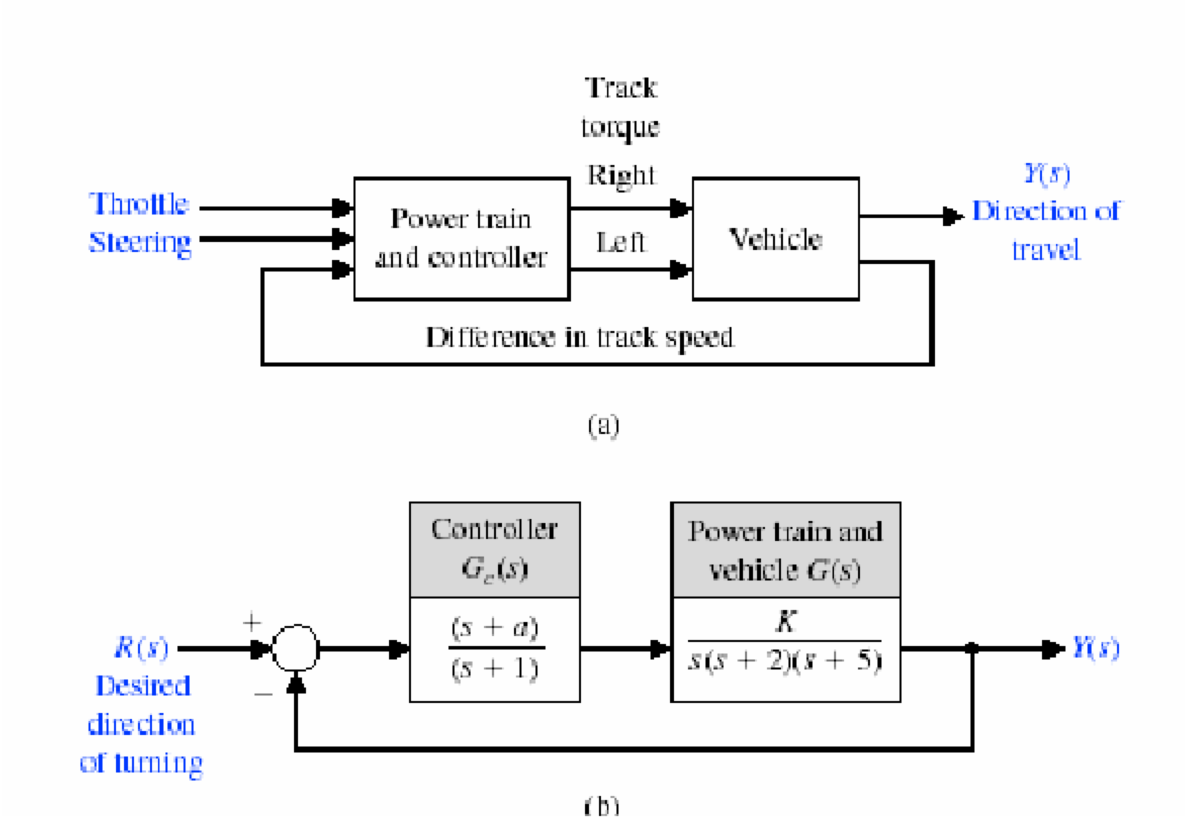
It is often necessary to know the relative damping of each root to the characteristic equation. Relative system stability can be measured by observing the relative real part of each root. In this diagram r2 is relatively more stable than the pair of roots labeled r1.



One method of determining the relative stability of each root is to use an axis shift in the s-domain and then use the Routh array as shown in Example 6.6 of the text.

Design Example: Tracked Vehicle Turning Control

Problem statement: Design the turning control for a tracked vehicle. Select K and a so that the system is stable. The system is modeled below.



Design Example: Tracked Vehicle Turning Control

The characteristic equation of this system is:

$$1 + G_c \cdot G(s) = 0$$

or

$$1 + \frac{K(s + a)}{s(s + 1)(s + 2)(s + 5)} = 0$$

Thus,

$$s(s + 1)(s + 2)(s + 5) + K(s + a) = 0$$

or

$$s^4 + 8s^3 + 17s^2 + (K + 10)s + Ka = 0$$

To determine a stable region for the system, we establish the Routh arra

s^4	1	17	Ka
s^3	8	$(K+10)$	0
s^2	b_3	Ka	
s^1	c_3		
s^0	Ka		

where

$$b_3 = \frac{126 - K}{8} \quad \text{and} \quad c_3 = \frac{b_3(K + 10) - 8Ka}{b_3}$$

Design Example: Tracked Vehicle Turning Control

s^4	1	17	Ka
s^3	8	$(K+10)$	0
s^2	b_3	Ka	
s^1	c_3		
s^0	Ka		

where

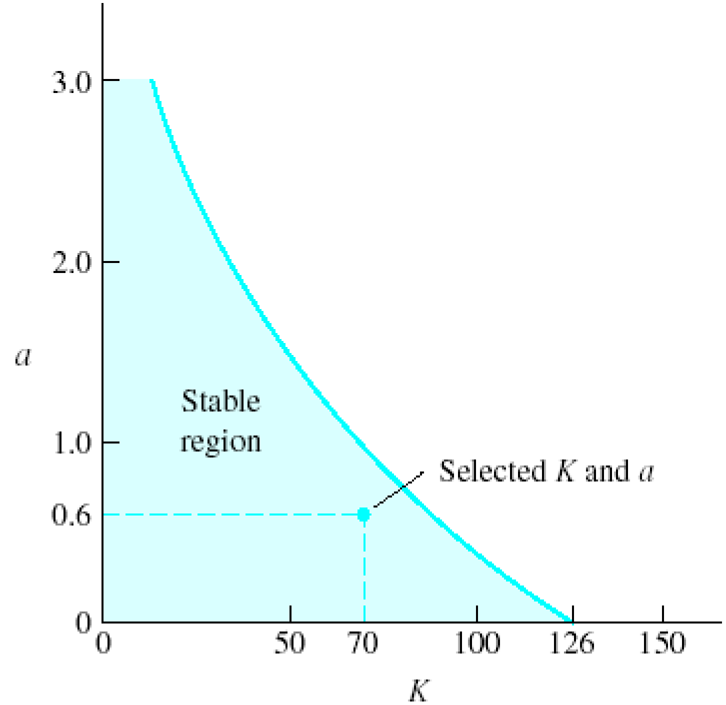
$$b_3 = \frac{126 - K}{8} \quad \text{and} \quad c_3 = \frac{b_3(K+10) - 8Ka}{b_3}$$

Therefore,

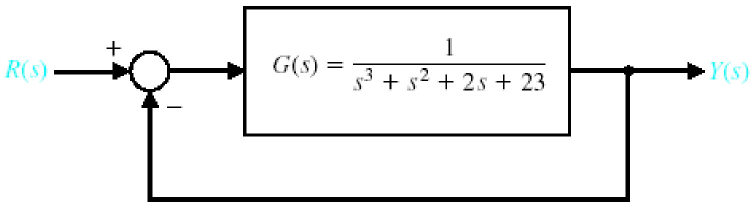
$$K < 126$$

$$K \cdot a > 0$$

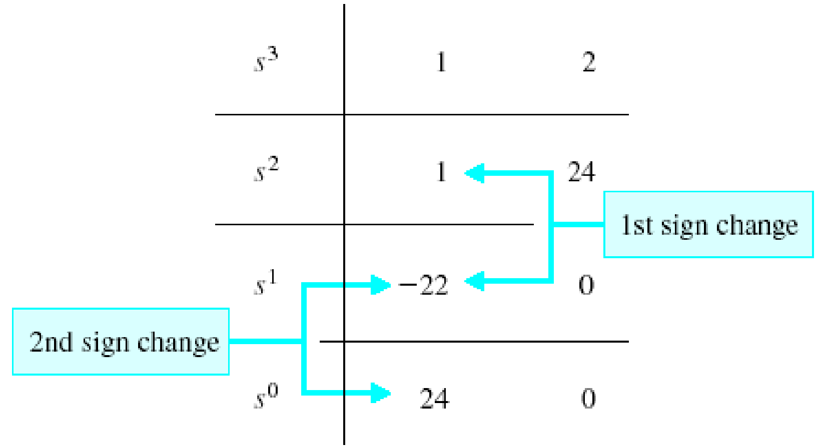
$$(K+10)(126-K) - 64Ka > 0$$



System Stability Using MATLAB



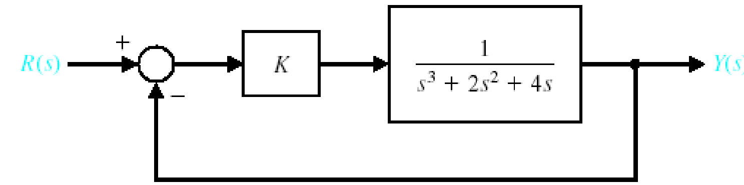
Closed-loop control system with $T(s) = Y(s)/R(s) = 1/(s^3 + s^2 + 2s$

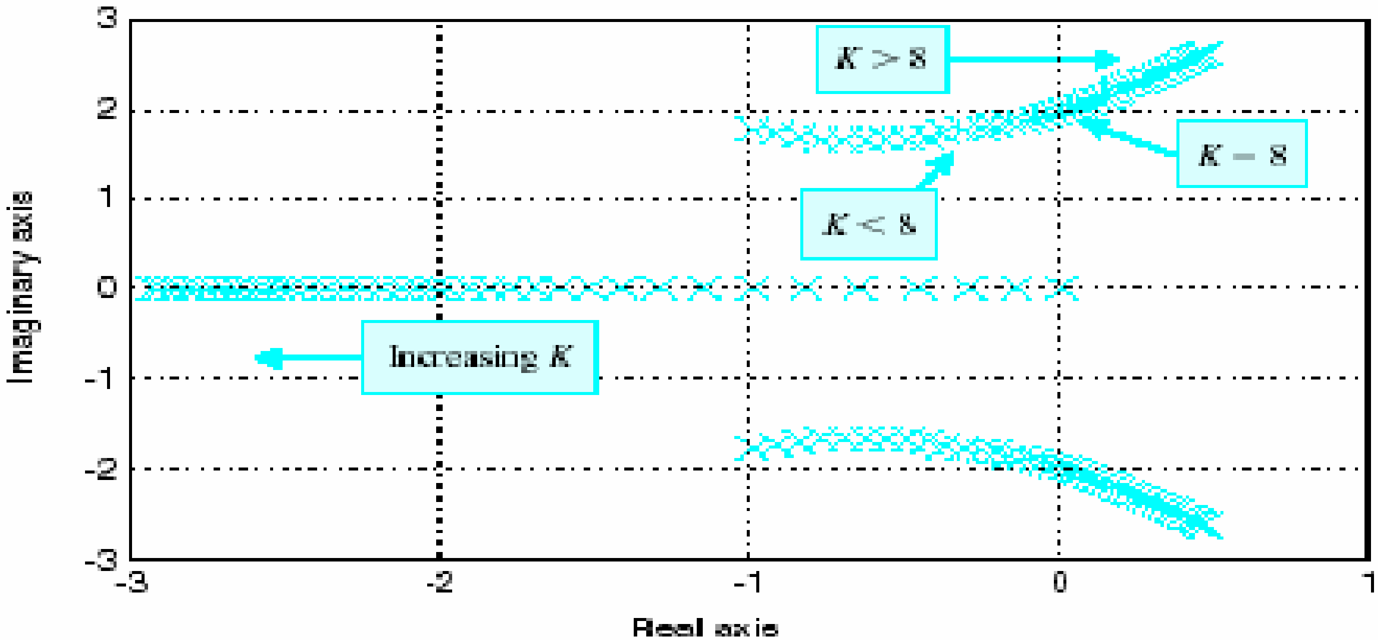



```
>>numg=[1]; deng=[1 1 2 23]; sysg=tf(numg,deng);
>>sys=feedback(sysg,[1]);
>>pole(sys)


ans =


   -3.0000
    1.0000 + 2.6458i
    1.0000 - 2.6458i
```

Unstable poles

System Stability Using MATLAB





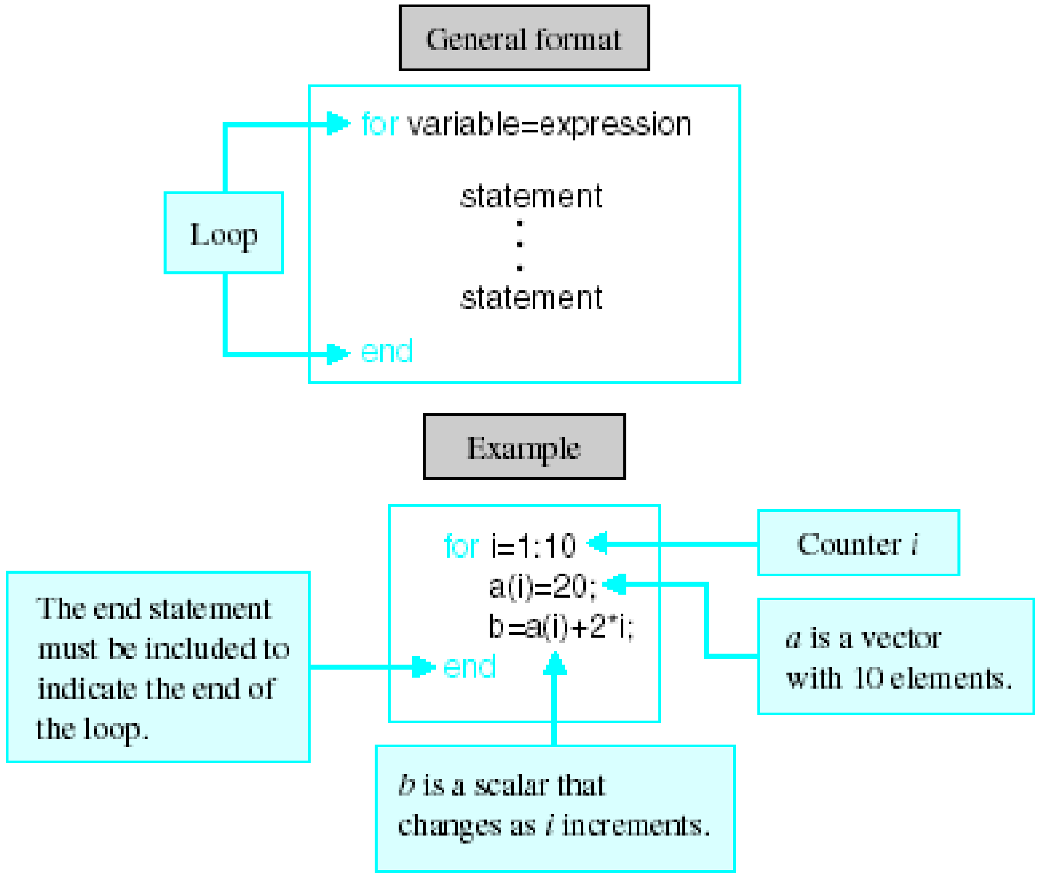
(a)

```
% This script computes the roots of the characteristic
% equation q(s) = s^3 + 2 s^2 + 4 s + K for 0<K<20
%
K=[0:0.5:20];
for i=1:length(K)
    q=[1 2 4 K(i)];
    p(:,i)=roots(q);
end
plot(real(p),imag(p),'x'), grid
xlabel('Real axis'), ylabel('Imaginary axis')
```

Loop for roots as a function of K

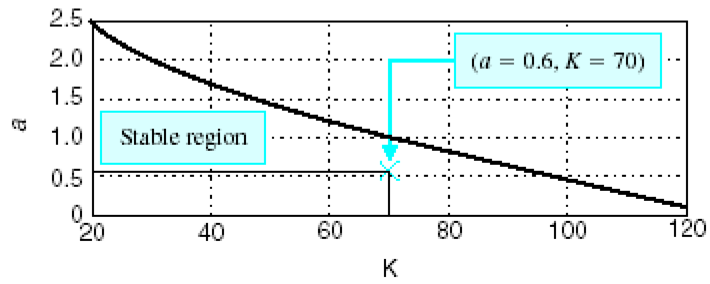
(b)

System Stability Using MATLAB



The for function and an illustrative example.

System Stability Using MATLAB



(a)

twotrackstable.m

```
% The a-K stability region for the two track vehicle
% control problem
%
a=[0.1:0.01:3.0]; K=[20:1:120];
x=0*K; y=0*K;
n=length(K); m=length(a);
for i=1:n
 for j=1:m
  q=[1, 8, 17, K(i)+10, K(i)*a(j)];
  p=roots(q);
  if max(real(p)) > 0, x(i)=K(i); y(i)=a(j-1); break; end
 end
end
plot(x,y), grid, xlabel('K'), ylabel('a')
```

Range of a and K

Initialize plot vectors as zero vectors of appropriate lengths.

Characteristic polynomial

For a given value of K: determine first value of a for instability.

(b)

(a) Stability region for a and K for two-track vehicle turning control.
(b) MATLAB script.